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Hopf bimodules are modules

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Abstract

We construct an algebra X associated to a finite-dimensional Hopf algebra A, such that there exists a vector space-preserving equivalence of categories between the categories of Hopf bimodules over A and of left X-modules. We show that X is isomorphic to the direct tensor product of the Heisenberg double of A and the opposite of its Drinfeld double. © 1998 Elsevier Science B.V. All rights reserved.

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1. Introduction

It is known from [10, 11] that Hopf bimodules over a finite-dimensional Hopf algebra A over a field k make up a braided category isomorphic to that of the representations of the Drinfeld double of A. The isomorphism of categories do not preserve the underlying vector space of the representation; moreover, while the braiding of Hopf bimodules concerns tensor products over A, the braiding for the representations of the Drinfeld double corresponds to tensor products over k. These facts appear as a preliminary of the principal result of [10]. We thank Montgomery for pointing out that these results were independently proved in [11].

Hopf bimodules were introduced by Nichols [9]. They were called bicovariant bimodules by Woronowicz who used them in [13]. These structures are fundamental to the classification of Hopf structures over path algebras, see [4].

The question arises to construct an algebra X such that the category of Hopf A-bimodules and the category of left X-modules are equivalent. It is well known that the

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Heisenberg double of A is an algebra, the representations of which makeup a category equivalent to the category of Hopf A-modules. We construct such an associative algebra X, which has a dimension of the fourth power of the dimension of A.

We provide two descriptions of the algebra X. The first is obtained by direct construction, the second is provided by Proposition 3.3, whose proof is based on the classification of Hopf bimodules and on Morita theory.

In order to modify the braided category of representations of the Drinfeld double, a twist of the product of X by means of a non-commutative 2-cocycle should be considered, see, for instance, [7] for the appropiate formalism. Both descriptions of X could be useful for this purpose.

2. The Heisenberg double

Recall that a left Hopf module M over a Hopf algebra A is in the first place a left A-module, which is a left A-comodule such that the structure map $\delta : M \longrightarrow A \otimes M$ is a morphism of left A-modules. The space $A \otimes M$ is equipped with the natural structure of left $A \otimes A$-module, and becomes a left A-module through the comultiplication of A.

Hopf modules form an abelian category, and the Heisenberg double $\mathscr{H}(A)$ is an algebra such that a left $\mathscr{H}(A)$-module is simultaneously a left A-Hopf module. The algebra $\mathscr{H}(A)$ is the crossed product $A^{*\mathrm{op}} \underline{\otimes} A$ under the exchange of the tensorands given below. We give a proof of this known fact (see [8] and references therein) in order to motivate the construction of the algebra X required in the introduction.

Notice at first that a left A-comodule M translates into a left $A^{*\mathrm{op}}$-module with structure map

$$A^{*\mathrm{op}} \otimes M \xrightarrow{1 \otimes \delta} A^{*\mathrm{op}} \otimes A \otimes M \xrightarrow{\mathrm{tr} \otimes 1} k \otimes M = M,$$

where tr is the trace or evaluation map.

Secondly, we point out that in the case of a Hopf module, the structure map above will be a left A-module morphism once A^* is equipped with a suitable left A-module structure, namely, the dual of the left regular representation reversed to the left using the antipode: for $a, b \in A$, $l \in A^*$ and S the antipode, we have $(al)(b) = l(S(a)b)$. Indeed, $1 \otimes \delta$ is a left A-module map since δ has this property, and tr is a left A-module map where k is the trivial A-module obtained with the co-unit.

Hence, a left A-Hopf module is given by a left A-module M affording simultaneously a left $A^{*\mathrm{op}}$-module such that the structure map of the last is an A-module map, which means that both actions are related by

$$a(lm) = \sum (a_{(1)}l)(a_{(2)}m).$$

Consequently, the algebra $A^{*\mathrm{op}} \underline{\otimes} A$ acts on M, where the algebras $A^{*\mathrm{op}}$ and A are subalgebras through the natural inclusions and

$$(1 \otimes a)(l \otimes 1) = \sum a_{(1)}l \otimes a_{(2)}.$$

Notice that this formula provides an associative algebra structure on the tensor product space $A^{*\mathrm{op}} \otimes A$, since it is given by the left $A^{*\mathrm{op}}$-module algebra structure on A. This exchange is a Yang–Baxter operator (see [3]), the produced algebra is the Heisenberg double $\mathscr{H}(A)$. Moreover, the procedure we have described for a Hopf module can be reversed using standard duality arguments since the representation spaces are finite dimensional, hence, the corresponding functors are inverses of one to the other.

Remark 2.1. It is known that the category of Hopf A-modules is semi-simple with only one simple object, namely A (see [1, 12]). Accordingly, the Heisenberg double is a simple algebra. The Wedderburn–Artin classification of semi-simple algebras compel it to be isomorphic to $\mathrm{End}_k H$, the simple matrix algebra of dimension $(\dim_k H)^2$ (see [8] for another proof).

3. The algebra X

Let A be an associative k-algebra. An A-bimodule is a vector space M with left and right A-module structures verifying $(am)b = a(mb)$ for $a, b \in A$ and $m \in M$. Now let A be a k-Hopf algebra. A bicomodule M is a vector space with left and right A-comodule structures $\delta_{\mathrm{L}} : M \to A \otimes M$ and $\delta_{\mathrm{R}} : M \to M \otimes A$, verifying $(1 \otimes \delta_{\mathrm{R}})\delta_{\mathrm{L}} = (\delta_{\mathrm{L}} \otimes 1)\delta_{\mathrm{R}}$. A Hopf bimodule over A is an A-bimodule equipped with a structure of an A-bicomodule such that both structure maps δ_{L} and δ_{R} are A-bimodule maps. The category they form is isomorphic to the category of modules of the Drinfeld double [6] of A, see [10]. We first notice that an A-bimodule structure translates into an $A^{*\mathrm{op}}$-bimodule; the opposite structure is required in order to have the left (resp. right) coaction issued by the left (resp. right) action. The structure map of this biaction is the following:

$$\phi : A^* \otimes M \otimes A^* \xrightarrow{1 \otimes \delta_{\mathrm{L}} \otimes 1} A^* \otimes A \otimes M \otimes A^* \xrightarrow{\mathrm{tr} \otimes 1 \otimes 1} k \otimes M \otimes A^*$$
$$= M \otimes A^* \xrightarrow{\delta_{\mathrm{R}} \otimes 1} M \otimes A \otimes A^* \xrightarrow{1 \otimes \mathrm{tr}^*} M \otimes k = M.$$

This map ϕ is an A-bimodule map, when A^* is endowed with either of the A-bimodule structures given below. At once, $\mathrm{tr} : A^* \otimes A \to k$ is an A-bimodule map, for the following A-bimodule structure on A^*:

$$al(x) = l(S(a)x) \quad \text{and} \quad la(x) = l(xS^{-1}(a)).$$

In order to also have $\mathrm{tr}^* : A \otimes A^* \to k$, an A-bimodule map, the preceeding structure has to be modified using the automorphisms S^2 and S^{-2} of A

$$a\bar{k}(x) = \bar{k}(S^{-1}(a)x) \quad \text{and} \quad \bar{k}a(x) = \bar{k}(xS(a)), \quad \text{for } \bar{k} \in A^*, \ a \in A.$$

Henceforth, the second A-bimodule structure will be denoted $\overline{A^*}$, and the map ϕ is an A-bimodule map

$$\phi : A^* \otimes M \otimes \overline{A^*} \xrightarrow{1\otimes\delta_L\otimes 1} A^* \otimes A \otimes M \otimes \overline{A^*} \xrightarrow{\mathrm{tr}\otimes 1\otimes 1} k \otimes M \otimes \overline{A^*}$$
$$= M \otimes \overline{A^*} \xrightarrow{\delta_R\otimes 1} M \otimes A \otimes \overline{A^*} \xrightarrow{1\otimes \mathrm{tr}^*} M \otimes k = M.$$

So, Hopf A-bimodules are A-bimodules having an extra A^*-bimodule structure such that the structure map $\phi : A^* \otimes M \otimes \overline{A^*} \to M$ is an A-bimodule map. One recovers the left and right coactions via the usual duality argument from the left and right actions described by ϕ, since the representation spaces are finite dimensional. This translates into

$$\phi(a(l \otimes m \otimes \bar{k})b) = a\phi(l \otimes m \otimes \bar{k})b, \sum (a_1 l b_1)(a_2 m b_2)(a_3 \bar{k} b_3) = alm\bar{k}b.$$

We have $a_1 l b_1(x) = l(S(a_1)xS^{-1}(b_1))$, while $a_3\bar{k}b_3\ (x) = \bar{k}(S^{-1}(a_3)xS(b_3))$.

Consider now the fact that A-bimodules and $A\otimes A^{\mathrm{op}}$-modules are the same, as well as A^*-bimodules and $A^* \otimes A^{*\mathrm{op}}$-modules, where these algebras are direct tensor products, i.e. the tensorands commutes. Under these identifications, ϕ becomes an $A \otimes A^{\mathrm{op}}$ left module map while considering $A^* \otimes M \otimes \overline{A^*}$ as a left $A \otimes A^{\mathrm{op}}$-module

$$\sum ((a_1 l b_1) \otimes (a_3 \bar{k} b_3))((a_2 \otimes b_2)m) = a \otimes b((l \otimes \bar{k})m).$$

Consider the algebra $(A^* \otimes A^{*\mathrm{op}}) \underline{\otimes} (A \otimes A^{\mathrm{op}})$ operating on Hopf A-bimodules where the first and last two tensorands keep the natural multiplication and

$$(1\otimes 1)\underline{\otimes}(a \otimes b)\ (l \otimes \bar{k})\underline{\otimes}(1 \otimes 1) = \sum (a_1 l b_1 \otimes a_3 \bar{k} b_3)\underline{\otimes}(a_2 \otimes b_2).$$

Finally, notice that

$$alb(x) = l(S(a)xS^{-1}(b)) = \sum l_1(S(a))l_2(x)l_3(S^{-1}(b))$$

by the definition of the comultiplication on A^*. Therefore,

$$alb = \sum l_1(Sa)l_3(S^{-1}b)l_2$$

and

$$a\bar{k}b = \sum k_1(S^{-1}a)k_3(Sb)k_2$$

using the described bimodule structures on A^*.

Theorem 3.1. *Let A be a finite-dimensional Hopf algebra. There is a vector space-preserving equivalence of categories between the category of left finite-dimensional modules over the Drinfeld double of A and the category of finite-dimensional representations of the product*

$$X = (A^* \otimes A^{*\mathrm{op}}) \underline{\otimes} (A \otimes A^{\mathrm{op}}),$$

where

$$(a \otimes b)(l \otimes k) = \sum l_1(Sa_1)k_1(S^{-1}a_3)l_3(S^{-1}b_1)k_3(Sb_3)(l_2 \otimes k_2) \otimes (a_2 \otimes b_2).$$

As noted, left modules over the Drinfeld double of A are equivalent to Hopf A-bimodules. The previous discussion shows that Hopf A-bimodules are equivalent to left X-modules, whence the theorem.

Remark 3.2. We have mentioned in the introduction that the category of A-Hopf bimodules is braided, and, in particular, monoidal. We infer the same property for the category of left X-modules, by structure transport through the category equivalences. However, the resulting tensor product of X-modules is over A, in other words the forgetful functor to vector spaces is not monoidal.

We wish to infer the abstract structure of the associative algebra X that we have obtained. For this purpose, the following is useful: by Theorem 4.1.1 of [12], any Hopf bimodule M is of the form $A \otimes C$, where A is the free rank one A-module and C is the space of left co-invariants. The equivalence between A-Hopf bimodules and modules over the Drinfeld double given in [10] shows that C is equipped with a right action of the Drinfeld double $\mathscr{D}(A)$ of A. Moreover, the morphism space between two Hopf bimodules is provided by the $\mathscr{D}(A)$-morphisms between their co-invariants.

First, we quote a well known fact from Morita theory. Let Λ be a finite-dimensional associative k-algebra and consider $\mathrm{End}_k V$ the simple algebra of endomorphisms of a finite-dimensional k-vector space V. A module over the direct tensor product of algebras $\mathrm{End}_k V \otimes \Lambda$ is isomorphic to $V \otimes C$, where V is the simple $\mathrm{End}_k V$-module and C is a Λ-module. Notice that, since $\mathrm{End}_k V$ is a simple algebra, any module is isotypic, isomorphic to a direct sum of copies of V. The morphisms between $V \otimes C$ and $V \otimes C'$ are identified with the space $\mathrm{Hom}_\Lambda(C, C')$ since, by Schur's lemma, $\mathrm{End}_{\mathrm{End}_k V} V = k$.

Proposition 3.3. *The algebra X constructed before is isomorphic to the direct tensor product of algebras $\mathscr{H}(A) \otimes \mathscr{D}(A)^{\mathrm{op}}$.*

In order to prove this proposition, we need the following result. We provide a proof for completeness.

Proposition 3.4. *Let Λ and Λ' be finite-dimensional algebras over a field k. Suppose there exists an equivalence of their module categories which preserves the k-dimension of the modules. Then, Λ and Λ' are isomorphic.*

We recall first some results of Morita theory (see, for instance, [2]). Consider $\mathscr{P}$ a family of projective indecomposable left Λ-modules such that any projective indecomposable left Λ-modules is isomorphic to exactly one element of $\mathscr{P}$. By the Krull–Schmidt theorem, $\mathscr{P}$ is a finite set.

For each finite-dimensional algebra Λ' Morita equivalent to Λ (i.e. such that the module categories are equivalent), there exist a sequence of strictly positive integers $a=(a_P)_{P\in\mathscr{P}}$ such that $\Lambda'\simeq \mathrm{End}_\Lambda(\mathscr{P}_a)^{\mathrm{op}}$, where $\mathscr{P}_a=\bigoplus_{P\in\mathscr{P}} a_P P$. Moreover, any equivalence of categories is isomorphic to the functor $\mathrm{Hom}_\Lambda(\mathscr{P}_a,-)$. Notice that, if $\Lambda=\mathrm{End}_k V$ is a simple algebra, then $\mathscr{P}=\{V\}$ and, for $a=a_V$, the corresponding algebra is $\mathrm{End}_{\mathrm{End}_k V}(aV)$, which is an $a\times a$ matrix algebra by Schur's lemma.

Proof of Proposition 3.4. Consider for Λ a set $\mathscr{P}$ as before. Let $\mathscr{S}$ be the corresponding set of simple left Λ-modules, $\mathscr{S}=\{P/\mathrm{rad}\,P \mid P\in\mathscr{P}\}$, where $\mathrm{rad}\,P$ is the unique maximal submodule of P. Each simple Λ-module is isomorphic to exactly one element of $\mathscr{S}$. Moreover, there is a bijection between $\mathscr{P}$ and $\mathscr{S}$, the inverse of the map $P\mapsto P/\mathrm{rad}\,P$ associates to each simple module $S\in\mathscr{S}$ its projective cover $P(S)$ (see [2, 5] for the proofs of these facts).

Let Λ' be an algebra Morita equivalent to Λ, so $\Lambda'\simeq\Lambda_a=\mathrm{End}_\Lambda(\mathscr{P}_a)^{\mathrm{op}}$ for $a=(a_P)_{P\in\mathscr{P}}$, and assume that

$$\dim_k \mathrm{Hom}_\Lambda(\mathscr{P}_a,M)=\dim_k M$$

for any Λ module M. We want to infer that Λ_a and Λ are isomorphic algebras. For each simple Λ-module $S\in\mathscr{S}$ we have

$$\begin{aligned}\mathrm{Hom}_\Lambda(\mathscr{P}_a,S)&=\mathrm{Hom}_\Lambda(\mathscr{P}_a/\mathrm{rad}\,\mathscr{P}_a,S)=\bigoplus_{P\in\mathscr{P}}\mathrm{Hom}_\Lambda(a_P(P/\mathrm{rad}\,P),S)\\ &=\bigoplus_{T\in\mathscr{S}}\mathrm{Hom}_\Lambda(a_{P(T)}T,S)=\mathrm{Hom}_\Lambda(a_{P(S)}S,S)=a_{P(S)}\mathrm{End}_\Lambda S.\end{aligned}$$

Consequently, $\dim_k\mathrm{Hom}_\Lambda(\mathscr{P}_a,S)=a_{P(S)}d_S$, where $d_S=\dim_k\mathrm{End}_\Lambda S$ (notice that $d_S\geq 1$ and that $d_S=1$ if k is algebraically closed).

Consider now $t=(t_P)_{P\in\mathscr{P}}$, the sequence of multiplicities of the indecomposable left projectives modules in the free rank-one Λ-module, i.e. Λ is isomorphic as a Λ-module to $\bigoplus_{P\in\mathscr{P}} t_P P$. It is well known that for each $S\in\mathscr{S}$ we have $\dim_k S = t_{P(S)}d_S$; indeed, consider the maximal semi-simple algebra quotient $\Lambda/\mathrm{rad}\,\Lambda=\bigoplus_{P\in\mathscr{P}} t_P(P/\mathrm{rad}\,P)=\bigoplus_{T\in\mathscr{S}} t_{P(T)}T$ and recall that $\Lambda/\mathrm{rad}\,\Lambda\simeq\mathrm{End}_{\Lambda/\mathrm{rad}\,\Lambda}(\Lambda/\mathrm{rad}\,\Lambda)^{\mathrm{op}}$. Since there are no homomorphisms other than zero between non-isomorphic simple modules, this algebra is isomorphic to $\times_{T\in\mathscr{S}}\mathrm{End}_{\Lambda/\mathrm{rad}\,\Lambda}(t_{P(T)}T)^{\mathrm{op}}$. Notice that through the identification of an algebra with its opposite endomorphism ring, an endomorphism f acts on a projective module as f operates on it. Consequently, $\mathrm{End}_{\Lambda/\mathrm{rad}\,\Lambda}(t_{P(S)}S)^{\mathrm{op}}$ is the only algebra of the product which has non-zero action on S. Hence, S is a module over this simple algebra isomorphic to $M_{t_{P(T)}}(\mathrm{End}_{\Lambda/\mathrm{rad}\,\Lambda}T)$, where $M_n(D)$ denotes a $n\times n$ matrix algebra over a skewfield D. For the latter, a simple module corresponds to a column and, hence, is of dimension $n\dim_k D$, so we have $\dim_k S=t_{P(S)}\dim_k(\mathrm{End}_{\Lambda/\mathrm{rad}\,\Lambda}S)$. Clearly, $\mathrm{End}_{\Lambda/\mathrm{rad}\,\Lambda}S=\mathrm{End}_\Lambda S$, so $\dim_k S=t_{P(S)}d_S$.

Since $d_S\neq 0$, we infer that $a_P=t_P$ for all $P\in\mathscr{P}$, the module $\mathscr{P}_a$ is the free rank-one module and the algebra $\Lambda'\simeq\Lambda_a=\mathrm{End}_\Lambda(\mathscr{P}_a)^{\mathrm{op}}$ is isomorphic to Λ. □

Proof of Proposition 3.3. We know that $\mathscr{H}(A)$ is isomorphic to $\mathrm{End}_k A$, so the left modules over the algebra $\mathscr{H}(A) \otimes \mathscr{D}(A)^{\mathrm{op}}$ are of the form $A \otimes C$, where C is a right $\mathscr{D}(A)$ module. We already pointed out that Hopf bimodules over A are exactly of this kind, and we notice that morphisms as modules over $\mathscr{H}(A) \otimes \mathscr{D}(A)^{\mathrm{op}}$ or as Hopf bimodules over A coincide. Consequently, the algebra X is Morita equivalent to $\mathscr{H}(A) \otimes \mathscr{D}(A)^{\mathrm{op}}$. Moreover, the k-dimension of the modules remains constant under the equivalence, and the Proposition 3.4 result asserts that the algebras are isomorphic. □

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